Numerical Analysis of Laminar Natural Convection Heat Transfer around Two Vertical Fins by a Spectral Finite Difference Method

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Abstract

A numerical solution is presented for the natural convection heat transfer from two vertical fins using a spectral finite difference method. Virtual distant boundary conditions for two bodies that are compatible with plume behavior and with an overall continuity condition are introduced. A boundary-fitted coordinate system is formed. Streamlines, isotherms, mean Nusselt numbers and drag & lift coefficients are presented for a variety of dimensionless parameters such as a Grashof number and a Prandtl number at a steady-state. Extensive effectiveness of a spectral finite difference method was established.

Keyword: *natural convection, vertical fin, spectral finite difference method, distant boundary.*

1. Introduction

Laminar natural convection has many engineering applications. In particular, heat transfer from vertical fins is one of the major concerns on heat exchangers. There are a great amount of analytical and experimental articles reported on natural convection heat transfer problem; for example, the flow around a circular cylinder, a sphere, a vertical or horizontal plate. Almost all the work has dealt with a simply-connected or a doubly-connected region. For most external natural convection heat transfer problems, the distant condition plays an important role. There are many articles reported on natural convection heat transfer around one body accounting plume behavior [2], [3]. Fujii [3] reported on numerical analysis of the natural convection plume above a horizontal line and point heat source in 1963. Mochimaru [3] introduced virtual distant boundary conditions that are compatible with plume behavior above uniformly heated horizontal circular cylinder and with an overall continuity condition.

An analytical object is two vertical fins located in a row. The surfaces of two fins are assumed to be maintained at a uniform high temperature. At this stage, it is necessary to introduce virtual distant boundary conditions. The virtual distant boundary conditions for two bodies, the virtual boundary conditions considering an adjacent body, and extensive effectiveness of a spectral finite difference method [1] are presented.

2. Analysis

Fluids are assumed to be incompressible and Newtonian. The flow and the thermal field are assumed to be two-dimensional and laminar. The coordinate system is nondimensionalized with respect to a reference-body length L^*. Dimensionless variables of time t^*, a temperature T^*, a stream function ψ^*, a vorticity ζ^* are defined as follows;

$$l = \frac{l^*}{L^*}, t = \frac{t^* \nu \sqrt{Gr}}{L^{*2}}, T = \frac{T^* - T_a^*}{T_w^* - T_a^*}, \psi = \frac{\psi^*}{\nu \sqrt{Gr}}, \zeta = \frac{\zeta^* L^{*2}}{\nu \sqrt{Gr}}$$

where Gr is a Grashof number; l, coordinate; ν, kinematic viscosity; T^*_a, an ambient temperature; T^*_w, a wall temperature.

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Grashof number *Gr* is defined as follows;

$$Gr \equiv \frac{L^{*3} ga(T_w^* - T_a^*)}{\nu^2}$$

where *g* is gravitational acceleration, and *a* is a coefficient of thermal expansion.

Under a Boussinesq approximation, the governing equations (a vorticity transport equation, an energy equation, the relation between vorticity and stream function) can be written in non-dimensional forms as follows:

$$J\frac{\partial \varsigma}{\partial t} + \frac{\partial(\varsigma,\psi)}{\partial(\alpha,\beta)} = \frac{1}{\sqrt{Gr}}\left\{\frac{\partial^2 \varsigma}{\partial \alpha^2} + \frac{\partial^2 \varsigma}{\partial \beta^2}\right\} + \frac{\partial(T,y)}{\partial(\alpha,\beta)} \tag{1}$$

$$J\frac{\partial T}{\partial t} + \frac{\partial(T,\psi)}{\partial(\alpha,\beta)} = \frac{1}{Pr\sqrt{Gr}}\left\{\frac{\partial^2 T}{\partial \alpha^2} + \frac{\partial^2 T}{\partial \beta^2}\right\} \tag{2}$$

$$J\varsigma + \frac{\partial^2 \psi}{\partial \alpha^2} + \frac{\partial^2 \psi}{\partial \beta^2} = 0 \tag{3}$$

$$J \equiv \frac{\partial(x,y)}{\partial(\alpha,\beta)}$$

where *J* is Jacobian, and *Pr* is a Prandtl number. Viscous dissipation has been neglected in the energy equation. Equations have been considered conformal mapping from the physical domain in the (x,y) plane to the analytical domain in the (α,β) plane. The mapping function is given by

$$x+iy = \frac{K'}{\pi}\left(Z + \frac{\mathrm{dn}(\xi+i\eta,k)\mathrm{cn}(\xi+i\eta,k)}{\mathrm{sn}(\xi+i\eta,k)} + \frac{\pi(\xi+i\eta)}{2K'K}\right), \quad \frac{\pi}{K'}(\xi+i\eta) = \alpha + i\beta \tag{4}$$

where *k*=0.1434, *K* and *K'* are complete elliptic integrals of the first kind with the complementary parameter $\sqrt{(1-k^2)}$ and $\sqrt{(1-k'^2)}$ respectively. *Z* is a zeta function of Jacobi.

The coordinates and the geometry are defined as shown in Fig. 1, where *L* is the fin length. Since the flow field and the thermal field are assumed to be symmetric with respect to the vertical y-axis, only the left half of the calculate object is considered in this paper. Let dimensionless fin length *L* be a unity.

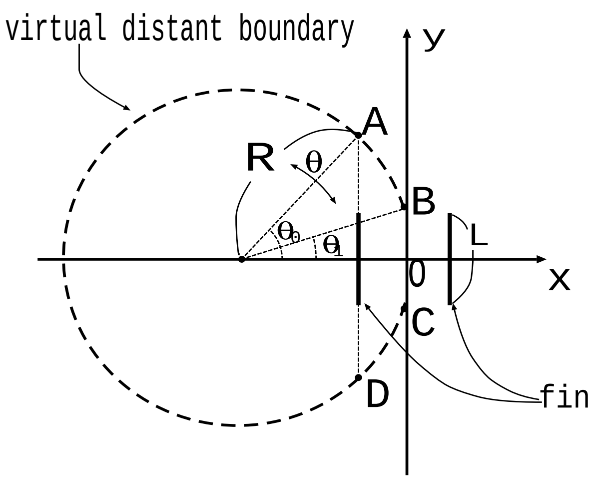


Fig. 1. Geometry and configuration

2.1 boundary conditions

Boundary conditions at the surface of the left fin are specified as follows:

$$\psi(\alpha_0,\beta)=\frac{\partial}{\partial\alpha}\psi(\alpha_0,\beta)=0 \tag{5}$$

$$T(\alpha_0,\beta)=1. \tag{6}$$

A part of non-slip boundary condition can be expressed in an implicit way by

$$\frac{2}{d^2}\psi(\alpha_0-d,\beta)+J\zeta(\alpha_0-d,\beta)=0 \tag{7}$$

where α_0-d is the coordinate of the grid point adjacent to the surface.

Virtual distant boundary conditions at a steady-state are introduced. Since the excess of a vorticity and temperature is remarkable in the plume, the behavior of a vorticity and temperature, a stream function can be expressed as

$$\xi=Gr^{1/5}R\,|\cos\theta_0-\cos\theta\,|\,(R\sin\theta_0+\delta)^{-2/5} \tag{8}$$

$$T_\infty=Gr^{-1/5}(R\sin\theta_0+\delta)^{-3/5}A^4h(A\xi) \tag{9}$$

$$\psi_\infty=Gr^{-3/10}(R\sin\theta_0+\delta)^{3/5}\sin\theta_0 Af(A\xi)H(\theta) \tag{10}$$

$$\varsigma_\infty=-Gr^{1/10}(R\sin\theta_0+\delta)^{-1/5}A^3f''(A\xi) \tag{11}$$

$$H(\theta)=\frac{\cos1.4\theta_0+\cot1.2\pi\sin1.4\theta_0}{\sin2\theta_0}\sin0.6\theta+\frac{\cos0.6\theta_0-\cot1.2\pi\sin0.6\theta_0}{\sin2\theta_0}\sin1.4\theta, \tag{12}$$

where the subscript ∞ means the virtual distant boundary, δ is a suitable parameter to be determined. And *f, h* are given by

$$f'''+\frac{3}{5}ff''-\frac{1}{5}f'^2+h=0 \tag{13}$$

$$h'+\frac{3}{5}Prfh=0, \tag{14}$$

with $h(0)=1,\ f(0)=f''(0)=0, h(\infty)=f'(\infty)=0$. The prime denotes differentiation with respect to $A\xi$. The parameter *A* is related with the mean Nusselt number Nu_m as

$$A^5Pr\int_0^\infty f'(\xi)h(\xi)d\xi=Nu_m. \tag{15}$$

At the upper right side of the left fin (from the point A to the point B in Fig. 1.), *H* is determined so as to be a unity. And, the lower right side of the left one (from the point C to the point D) is expressed through supposition of a uniform flow. The stream function near to the y-axis (from the point B to the point C) is constant considering existence of the right side fin. On this virtual boundary, heat flux is specified to be zero, and the vorticity is

vanished. Zero heat flux boundary conditions are considered at the virtual boundary from C to D too.

$$H(\theta)=1 \qquad \text{at} \quad \overline{AB} \tag{16}$$

$$\psi = Gr^{-3/10}(R\sin\theta_0+\delta)^{3/5}\sin\theta_0 Af(A\hat{\xi}) \qquad \text{at} \quad \overline{BC} \tag{17}$$

$$\frac{\partial T}{\partial \alpha}=0 \qquad \text{at} \quad \overline{BC} \text{ and } \overline{CD} \tag{18}$$

$$\zeta=0 \qquad \text{at} \quad \overline{BC} \text{ and } \overline{CD} \tag{19}$$

$$\psi = Gr^{-3/10}(R\sin\theta_0+\delta)^{3/5}\frac{\cos\theta-\cos\theta_0}{\cos\theta_1-\cos\theta_0}\sin\theta_0 Af(A\hat{\xi}) \qquad \text{at} \quad \overline{CD} \tag{20}$$

where $\hat{\xi}$ is a value of independent variable when an argument $\theta=\theta_1$.

At a partial Neumann-type virtual thermal boundary conditions Eqs. (18), a following equation can be used.

$$\frac{\partial}{\partial\alpha}T(\alpha_j,\beta,t)\cong\frac{1}{(2+c)d}\left\{T(\alpha_j,\beta,t)+cT(\alpha_j-d,\beta,t)-(1+c)T(\alpha_j-2d,\beta,t)\right\} \tag{21}$$

where d is a grid size near virtual boundary, and c is a constant chosen arbitrarily[4].

2.2 Numerical procedure

Let ψ, ζ, T be expressed as

$$\psi=\sum_{k=0}^{\infty}\psi_{ck}\cos k\beta+\sum_{k=1}^{\infty}\psi_{sk}\sin k\beta \tag{22}$$

$$\zeta=\sum_{k=0}^{\infty}\zeta_{ck}\cos k\beta+\sum_{k=1}^{\infty}\zeta_{sk}\sin k\beta \tag{23}$$

$$T=\sum_{k=0}^{\infty}T_{ck}\cos k\beta+\sum_{k=1}^{\infty}T_{sk}\sin k\beta \quad . \tag{24}$$

Substituting Eqs. (22) - (24) for governing equations, and decomposing the equations into cosine terms and sine terms gives the forms as

$$\sum_{k=0}^{\infty}f_{ck}(\alpha,t)\cos k\beta+\sum_{k=0}^{\infty}f_{sk}(\alpha,t)\sin k\beta=0$$

which reduces to

$$f_{ck}(\alpha,t)=0\,,\quad f_{sk}(\alpha,\mathrm{t})=0\,,\qquad k\geq 0\ .$$

Then each Fourier component in β can be separated from Eqs. (1) - (3). Hereafter subscripts *ck*, *sk* mean a wave number of cosine term, and a wave number of sine term respectively.

All second-order derivatives with respect to α are replaced by three-point central difference approximation. First-order derivatives with respect to α near boundary surface are used three-point forward difference

formulae. Time integrations are replaced by semi-implicit method. With these, the differential equations are converted into a set of algebraic equations. These equations are then solved iteratively per wave number.
Time difference approximation;

$$\frac{\partial}{\partial t} f(\alpha, t_i) = \frac{f(\alpha, t_{i+1}) - f(\alpha, t_i)}{\Delta t} \tag{25}$$

Spatial difference approximation;

$$\left.\frac{\partial f}{\partial \alpha}\right|_{\alpha=\alpha_k} = \frac{-f_{k-1} d_k}{d_{k-1}(d_k + d_{k-1})} + \frac{f_k (d_k - d_{k-1})}{d_{k-1} d_k} + \frac{f_{k+1} d_{k-1}}{d_k (d_k + d_{k-1})} \tag{26}$$

$$\left.\frac{\partial^2 f}{\partial \alpha^2}\right|_{\alpha=\alpha_k} = \frac{2 f_{k-1}}{d_{k-1}(d_k + d_{k-1})} - \frac{2 f_k}{d_{k-1} d_k} + \frac{f_{k+1}}{d_k (d_k + d_{k-1})} \tag{27}$$

To support high Grashof number cases, non-uniform grid spacing is used.

$$\alpha_k = \alpha_0 - c\left\{\frac{\sinh(k-1)\gamma}{\sinh \gamma} + 1\right\} \tag{28}$$

where α_k is a coordinate at the *k*-th grid point counted from the wall along α coordinate, γ is a suitable constant and *c* is a constant to be determined.

2.3 Dimensionless force

Dimensionless force acting on a fin surface along positive x direction is defined as a drag coefficient C_D,. And, dimensionless force acting on a fin surface along positive y direction is defined as a lift coefficient C_L

$$C_D = \frac{1}{\sqrt{Gr}} \int_{-\pi}^{\pi} \left(\varsigma \frac{\partial x}{\partial \beta} + \frac{\partial \zeta}{\partial \alpha} y \right) d\beta \tag{29}$$

$$C_L = \frac{-1}{\sqrt{Gr}} \int_{-\pi}^{\pi} \left(\varsigma \frac{\partial y}{\partial \beta} - \frac{\partial \zeta}{\partial \alpha} x \right) d\beta \tag{30}$$

A mean Nusselt number Nu_m at the surface of the left heat source is defined as follows:

$$Nu_m = -\pi \frac{\partial T_{c0}}{\partial \alpha} \tag{31}$$

2.4 Convergence criteria

To seek a steady-state solution, the iterative procedure is terminated when the following convergence criteria Eqs. (32), (33) are satisfied at the surface of the fin.

$$\left| \frac{C_{D(\mathrm{new})} - C_{D(\mathrm{old})}}{C_{D(\mathrm{new})}} \right| < 10^{-6} \tag{32}$$

$$\left| \frac{C_{L(\mathrm{new})} - C_{L(\mathrm{old})}}{C_{L(\mathrm{new})}} \right| < 10^{-6} \tag{33}$$

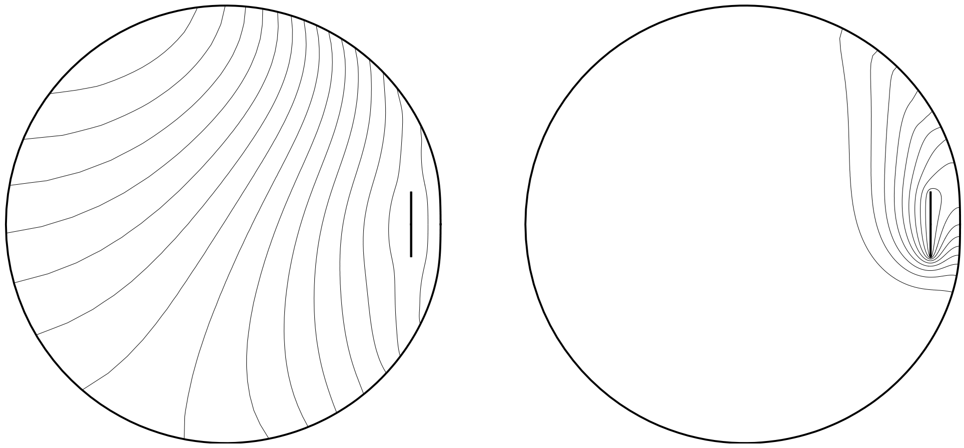
$$\left| \frac{Nu_{m(new)} - Nu_{m(old)}}{Nu_{m(new)}} \right| < 10^{-6} \tag{34}$$

where the subscripts (new) and (old) denote the current value and the value in the past respectively.

The number of partitions along α, and the wave number to seek a steady-state solution is determined so that the variation of dimensionless forces does not exceed one percent. In this paper, the number of partitions along α, and the wave number are used in the range of 100 to 200, and in the range of 30 to 40, respectively.

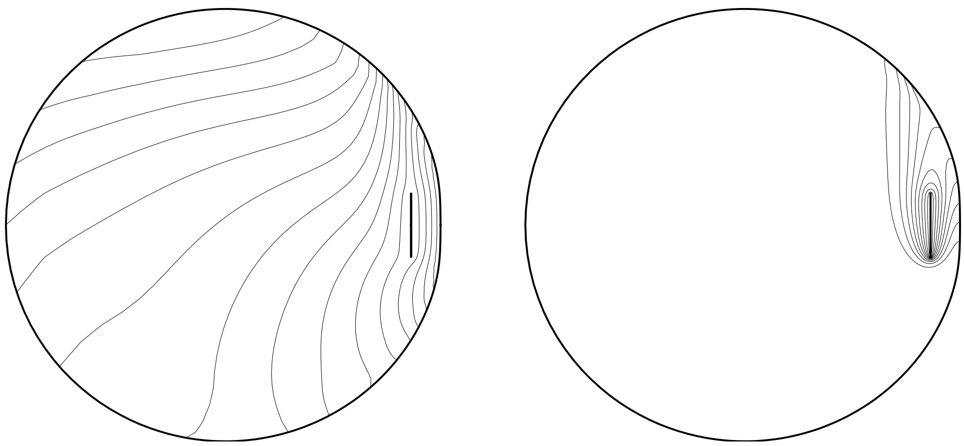
3. Results and discussion

A numerical solution for steady-state natural convection heat transfer accounting plume behavior around two vertical fins whose temperatures are maintained at a higher temperature than that of the ambient has been obtained, against a Rayleigh number in the range of 0.7 to 70000 for air (Pr=0.7) and water (Pr=7).



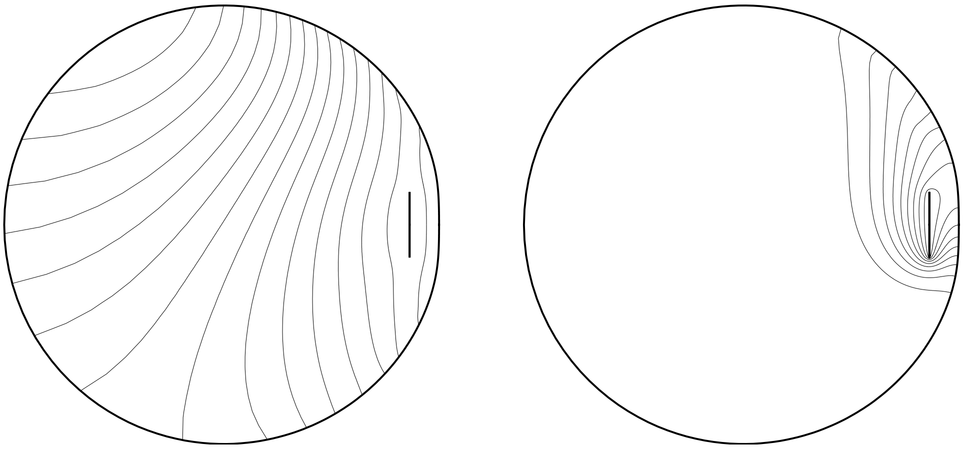
(a) (b)

Fig. 2. (a) Steady-state streamlines; $\Delta\psi$=0.24, (b) isotherms; ΔT=0.1 at Pr=0.7, Gr=1



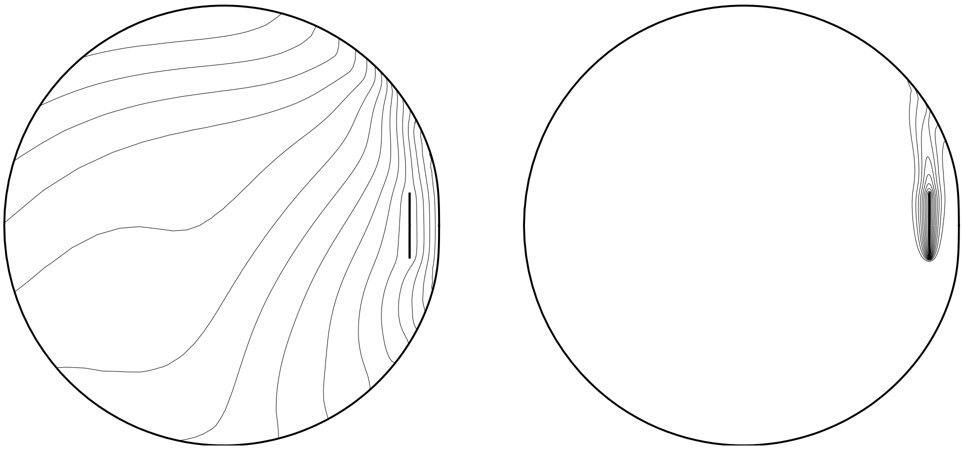
(a) (b)

Fig. 3. (a) Steady-state streamlines; $\Delta\psi$=0.07, (b) isotherms; ΔT=0.1 at Pr=0.7, Gr=1000



(a) (b)

Fig. 4. (a) Steady-state streamlines; $\Delta\psi = 0.135$, (b) isotherms; $\Delta T = 0.1$ at $Pr = 7$, $Gr = 1$



(a) (b)

Fig. 5. (a) Steady-state streamlines; $\Delta\psi = 0.035$, (b) isotherms; $\Delta T = 0.1$ at $Pr = 7$, $Gr = 1000$

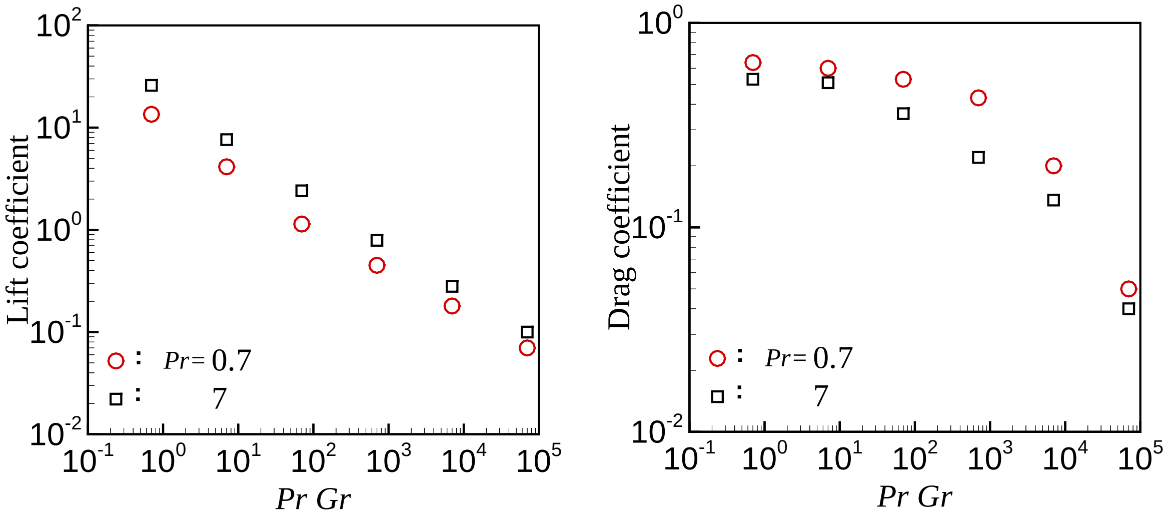


Fig. 6. (a) A lift coefficient C_L, (b) a drag coefficient C_D against $PrGr$

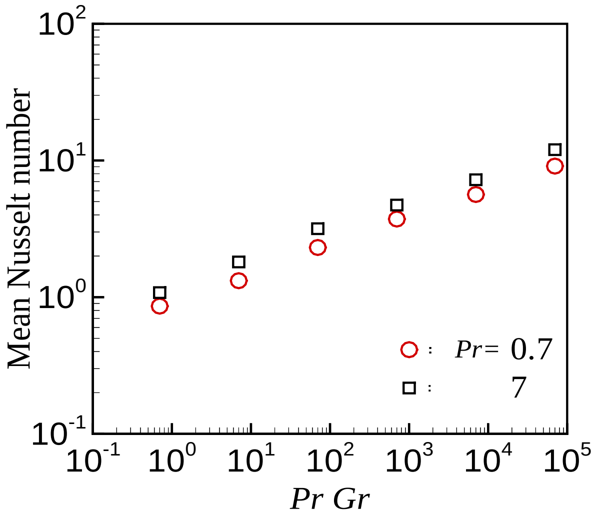


Fig. 7. A mean Nusselt number Nu_m against $PrGr$

Steady-state streamlines and isotherms are shown in Fig. 2-5. Thermal fields similar to solid heat conduction are obtained in case of a low Grashof number (see Fig. 2 (b)). A steady-state buoyancy plume above the fin is formed as an increase of a Grashof number.

The lift coefficient and the drag coefficient at a steady-state for various Grashof numbers and Prandtl numbers are presented in Fig. 6. The mean Nusselt number of the present at a steady-state is proportional to a Grashof number to the power 0.2(see Fig. 7). Clearly, the mean Nusselt number of water always indicates higher heat transfer rate than that of air, and the lift coefficient of water within the results of this investigation is bigger than that of air in case of a same Rayleigh number. However, the drag coefficient of water is smaller than that of air in case of a same Rayleigh number. The mean Nusselt number and the lift coefficient are in good agreement with the solution of the boundary layer equations.

4. Conclusion

Steady-state laminar natural convection heat transfer around two vertical fins using the virtual distant boundary conditions and the virtual boundary conditions was studied numerically by a spectral finite difference method for various Grashof numbers and Prandtl numbers. The virtual distant boundary conditions and the virtual boundary conditions for two bodies could demonstrate plume behavior very well. Extensive effectiveness of a spectral finite difference method was established.

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